

Section 3

DYNAMICS OF A RIGID BODY

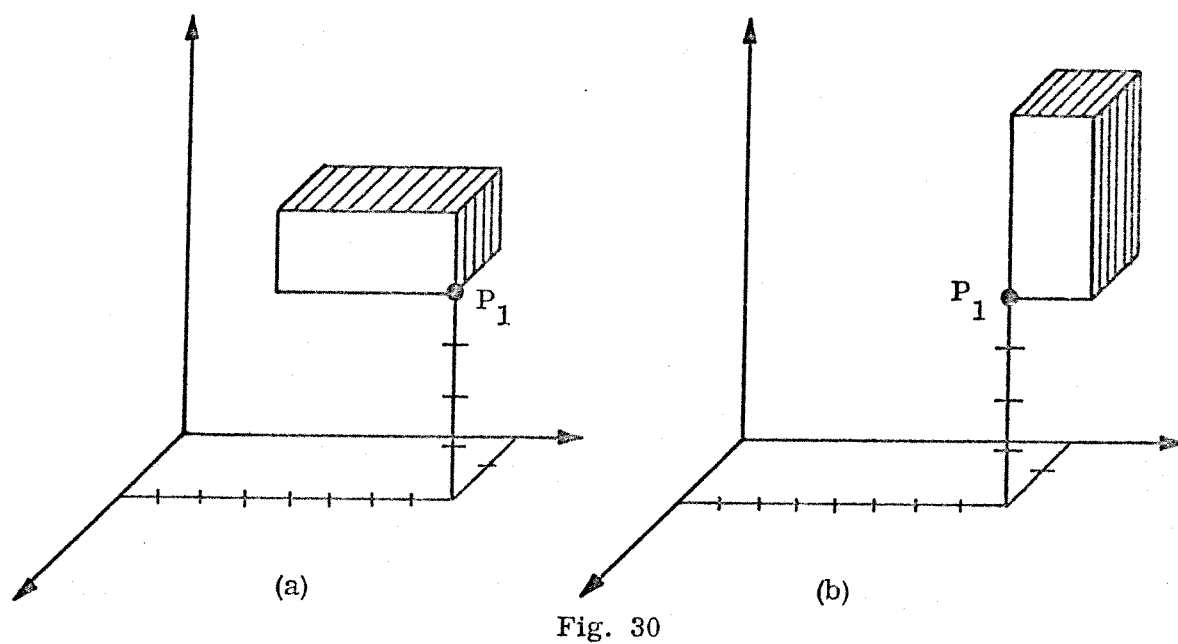
3.1 INTRODUCTION AND DEFINITION.

3.1.1 In general, the methods of particle dynamics are insufficient to adequately describe the motion of a physical body. In this section, an idealized but useful treatment of rigid-body motion is developed and extended from the concepts of particle motion presented previously. The definition of a rigid body is of paramount importance; one must constantly bear in mind that the theorems and laws developed in this section are fundamentally dependent on this definition.

A rigid body is defined as a grouping of particles of masses $m_1, m_2, m_3, \dots, m_n$ such that the distance between any two of them does not change with time. Note that the possibility of internal forces acting within the body is not precluded. Rather, one can infer from the definition that the internal forces are such that the distance between any two points remains invariant with respect to time. Thus, deformations, such as those due to stresses and strains, are not considered in this presentation. Furthermore, it is assumed that the rigid body can move freely in space according to the forces impressed on it; physical constraints on its motion are not considered.

3.2 INDEPENDENT PARAMETERS.

3.2.1 We may now raise the question: How many independent parameters are required to specify the instantaneous position and orientation of a rigid body with respect to a set of coordinate axes? Obviously, knowing the location (x_1, y_1, z_1) of one particle P_1 in the body is insufficient. The location of one point would give some information concerning the distance between the body and the origin of the coordinate axes but no indication of the orientation of the body. Figure 30 shows two possible positions of a rigid body which has one point fixed with respect to the axes.



Nor would the locations of two points P_1 and P_2 (Fig. 31) indicate the orientation of the body.

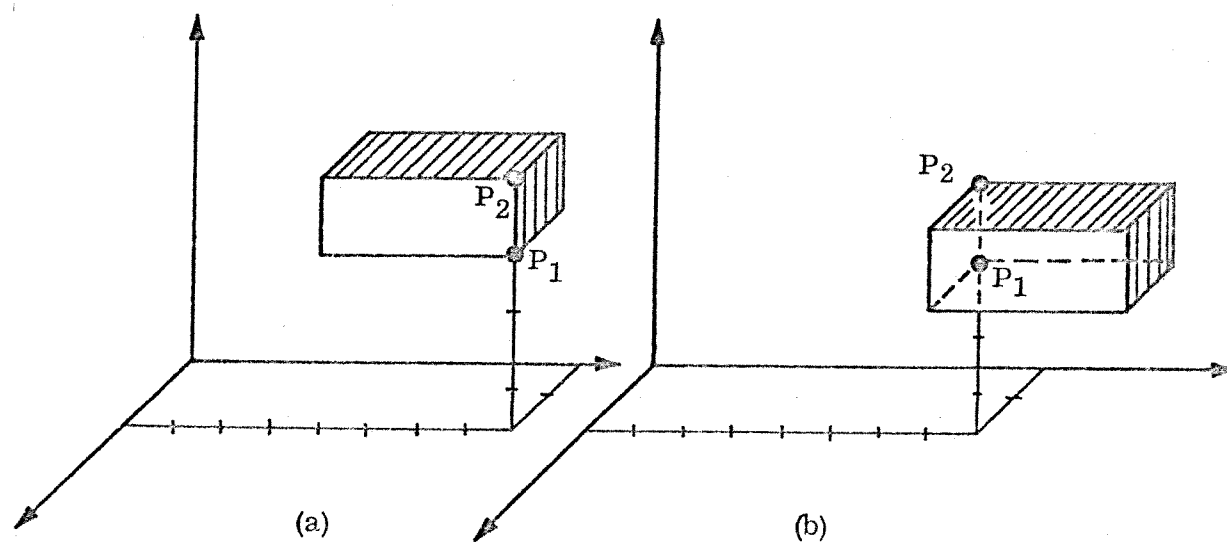


Fig. 31

However, suppose the instantaneous locations of three points P_1 , P_2 , and P_3 of the body were known: The distances between these points (Fig. 32) are fixed by definition.

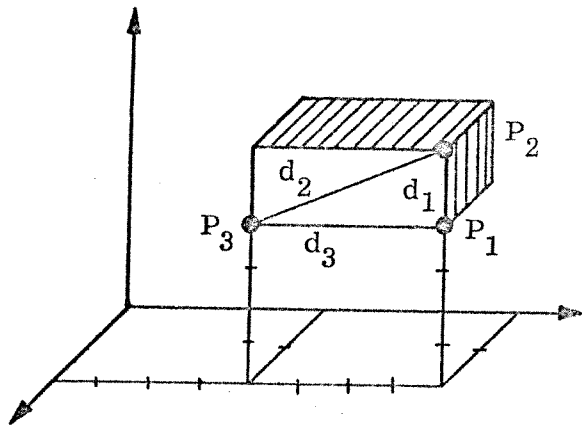


Fig. 32

Let d_1 be the distance between P_1 and P_2 , d_2 the distance between P_2 and P_3 , and d_3 the distance between P_1 and P_3 . Then the coordinates of P_1 , P_2 and P_3 , namely (x_1, y_1, z_1) , (x_2, y_2, z_2) , and (x_3, y_3, z_3) , respectively, would suffice to specify the position and orientation of the body at that time. Note, though, that the orientation is known only if the three points are not colinear.

These nine coordinates are not independent, as we shall see; that is, the body does not have nine independent degrees of freedom of motion. For example, three independent parameters are required to specify the location of one point, say P_1 . However, since the distance d_1 between P_1 and P_2 remains fixed (by definition of rigid body), P_2 is thus located on a sphere of radius d_1 about P_1 as center. Hence, only two independent parameters are required to locate P_2 , once P_1 is specified. Similarly, the third point P_3 is on a sphere of radius d_3 about P_1 and on another sphere of radius d_2 about P_2 . From which, we see that P_3 is on the circle of intersection of both spheres, and, therefore, only one independent parameter is required to locate it after the locations of P_1 and P_2 have been determined. A maximum of six independent parameters are thus required to specify the location of the three points and, hence, the position and orientation of the rigid body to which they are attached. Six independent parameters are required when the body can move freely in space subject to forces acting on it. Fewer independent parameters may suffice when physical constraints are placed on the motion.

One may also conceive of these six parameters as being divided into two groups: three parameters are required to specify the location of one point in the body with respect to some set of axes, and three parameters are required to specify the orientation of the body with respect to that point. The body is then said to have six degrees of freedom of motion.

In Fig. 33, X'_0 , Y'_0 , and Z'_0 are the axes of an inertial coordinate system with origin O_0 . Let X_0 , Y_0 , and Z_0 be another Cartesian coordinate system whose axes always remain parallel to the respective axes X'_0 , Y'_0 , and Z'_0 , but whose origin O is a point of the rigid body. The X_0 , Y_0 and Z_0 axes will be called the translating reference system. Let X , Y , and Z be a Cartesian coordinate system rigidly attached to the body, whose origin O is the same as that of the translating reference system; these axes will be known as the body axes, and, of course, will rotate with the body. The configuration of the rigid body B can then be described with respect to the body axes X , Y , and Z .

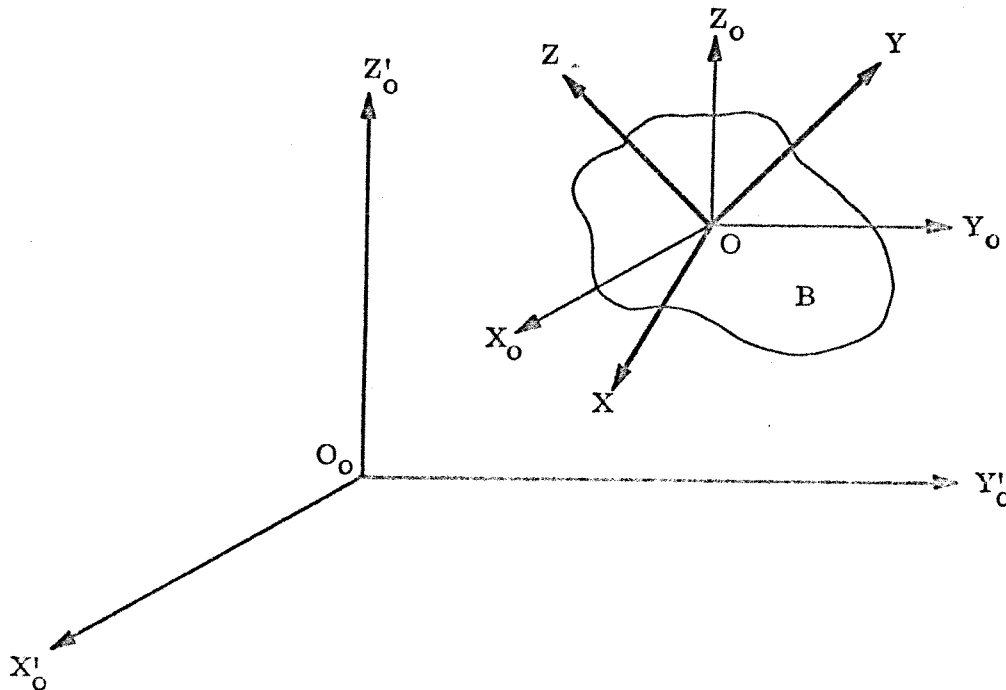


Fig. 33

The motion of the body is composed of:

- The translation of the origin O of the body axes
- The rotation of the body axes X , Y , and Z about the translating reference frame (X_o, Y_o, Z_o)

Thus, it is necessary to find a method which gives the instantaneous orientation of the X , Y , and Z axes with respect to the X_o , Y_o , and Z_o axes. For the moment, however, we turn to the translation problem; that of specifying the motion of the origin of the body axes.

3.3 LINEAR MOTION.

3.3.1 Returning to the original concept of a rigid body (i. e., an assemblage of particles of mass $m_1, m_2, m_3, \dots, m_n$), let $\bar{r}_1, \bar{r}_2$, and $\bar{r}_3$ be the corresponding radius vectors of the points from the origin of some inertial coordinate system. Let $(\bar{F}_{ext})_i$ and $(\bar{F}_{int})_i$ represent the sums of all external and internal forces acting on the i^{th} particle, where the internal forces are attributed to attractions and repulsions of other particles. Then, Newton's second law gives

$$m_i \ddot{\bar{r}}_i = (\bar{F}_{ext})_i + (\bar{F}_{int})_i \quad (3.1)$$

Summing over all the particles of the body,

$$\sum_i m_i \ddot{\bar{r}}_i = \sum_i (\bar{F}_{ext})_i + \sum_i (\bar{F}_{int})_i \quad (3.2)$$

Since the internal forces are due to attractions and repulsions between the particles and are not the result of external causes, Newton's third law holds; that is, if two particles exert forces on each other, the force exerted by the first on the second is equal to and opposite to the force exerted by the second on the first. The sum of all the internal forces must then be zero, and the summation equation becomes

$$\sum_i m_i \ddot{\bar{r}}_i = \sum_i (\bar{F}_{ext})_i \quad (3.3)$$

It is now convenient to define a point in the body known as the center of mass. The position vector $\bar{\mathbf{r}}$ of the center of mass with respect to some arbitrary coordinate system is given by

$$\bar{\mathbf{r}} = \begin{bmatrix} x_{cm} \\ y_{cm} \\ z_{cm} \end{bmatrix} = \frac{\sum_i m_i \bar{\mathbf{r}}_i}{\sum_i m_i} = \frac{\sum_i m_i \bar{\mathbf{r}}_i}{m} = \frac{1}{m} \begin{bmatrix} \sum_i m_i x_i \\ \sum_i m_i y_i \\ \sum_i m_i z_i \end{bmatrix} \quad (3.4)$$

where m is the total mass $\sum_i m_i$. Note that if the origin of the coordinate system is taken to be the center of mass itself, then the coordinates are $(0, 0, 0)$. Therefore,

$$\sum_i m_i \bar{\mathbf{r}}_i = 0 \quad (3.5)$$

Differentiating Eq. (3.4) twice with respect to time, we obtain

$$\ddot{\bar{\mathbf{r}}} = \frac{1}{m} \frac{d^2}{dt^2} (\sum_i m_i \bar{\mathbf{r}}_i)$$

and, since each particle is considered to have constant mass m_i ,

$$\ddot{\bar{\mathbf{r}}} = \frac{\sum_i m_i \ddot{\bar{\mathbf{r}}}_i}{m}$$

Substituting from Eq. (3.3) and multiplying by m , we have

$$m \ddot{\bar{\mathbf{r}}} = \sum_i (\bar{\mathbf{F}}_{ext})_i \quad (3.6)$$

Thus, the mass center of a system of particles moves as if all the mass were concentrated at that point and as if all external forces acted there. Hence, the component equations of Eq. (3.6) are the three equations of motion. It is important to realize that we are not saying that all the forces actually apply at the center of mass. All that is claimed is that the motion of the center of mass can be determined by considering that all forces act there.

3.4 ANGULAR VELOCITY.

3.4.1 We now discuss the fundamental concepts of angular motion. The angular velocity of the body axes about the translating axes is the same as the angular velocity of the particles in the body rotating about some instantaneous axis. Since the angular velocity of a particle is a vector (paragraph 2.10.1), the angular velocity of the body will also be a vector. The actual proof that it is a vector is beyond the scope of this report; however, we can intuitively see the truth of this fact. For example, if the body is moving such that its X-Z plane is rotating about its Y axis, the angular velocity vector lies along the Y axis. If the body is instantaneously rotating about any line, the instantaneous angular velocity will be along this line and will have direction according to the sense of the rotation. If applied forces were to cause the body to rotate about several axes of rotation simultaneously, the body angular velocity vectors about each line of rotation can be added vectorially into one total (resultant) angular velocity vector $\bar{\omega}$. Since any vector can be resolved into components with respect to any coordinate system, we can resolve the body angular velocity along the body axes. The components along the body axes X, Y, and Z are generally referred to as p, q, and r respectively; occasionally, ω_x , ω_y , ω_z are used. Thus,

$$\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k}$$

3.5 TORQUE AND ANGULAR MOMENTUM.

3.5.1 In seeking the relationship between the angular velocity of a body and the rotation caused by application of external forces, one intuitively feels that the rotation is dependent on

the magnitude and direction of each force and the distance from the point of application to the center of mass. The rotation-causing force is related to a concept known as torque. The resulting rotation and associated angular velocity are involved in a concept known as angular momentum. A relation connecting the cause (torque) and the effect (angular momentum) called the moment equation is developed in subsequent paragraphs. First, however, it is necessary to discuss the concepts of torque and angular momentum.

The torque $\bar{L}_P$ about point P exerted by force $\bar{F}$ at point Q is given by the cross-product of distance $\bar{PQ}$ and force $\bar{F}$. In other words,

$$\bar{L}_P = \bar{r} \times \bar{F} \quad (3.7)$$

where $\bar{r} = \overline{PQ}$.

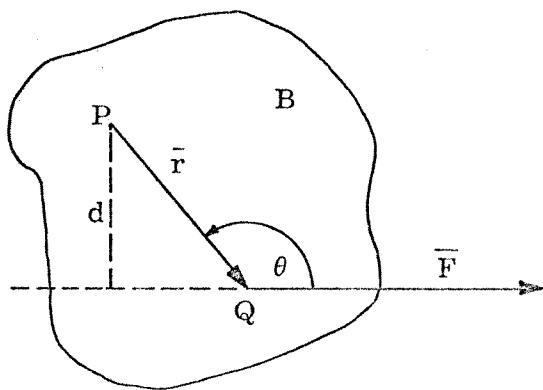


Fig. 34

The magnitude of the torque is given by $rF \sin \theta = Fd$, where d is the perpendicular distance from P to the line in which the force acts. Torque can then be con-

sidered as a force-moment. If P is fixed and both P and Q are on a planar body B of uniform thickness, then the force shown in the figure would cause the body to rotate about P in a counterclockwise direction. Thus, the torque vector would be parallel to and in the same direction as the angular velocity vector resulting from the application of only this one force.

The angular momentum $\bar{H}_P$ of a particle of mass m about point P is defined as the cross product of distance $\bar{r}$ from P to the particle and the linear momentum $m \dot{\bar{r}}$ of the particle:

$$\bar{H}_P = \bar{r} \times m \dot{\bar{r}} \quad (3.8)$$

Since the angular momentum is the moment of the linear momentum about P , it is also frequently called the moment of momentum. (We define the moment of a vector about a point as the cross product of the distance from the point to the origin of the vector with the vector itself. In this case, one may consider the vector to be localized instantaneously along a specific line of action).

The angular momentum of a rigid body about point P is the summation of the angular momenta of all the particles of the body:

$$\bar{H}_P = \sum_i (\bar{r}_i \times m_i \dot{\bar{r}}_i) \quad (3.9)$$

3.6 COMPONENTS OF ANGULAR MOMENTUM AND MOMENTS OF INERTIA.

3.6.1 Before determining the instantaneous rotation of the rigid body caused by forces, we need to find expressions for the components along the body axes of the angular momentum of the body with respect to its center of mass.

Let $\bar{i}$, $\bar{j}$, and $\bar{k}$ be the orthonormal vectors of the body coordinate system (X, Y, Z) which is rotating with angular velocity $\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k}$ with respect to the translating reference system (X_o, Y_o, Z_o) ; the origin of both systems is the body center of mass CM. Suppose P_i is a particle of mass m_i in the body at a distance $\bar{r}_i = x_i\bar{i} + y_i\bar{j} + z_i\bar{k}$ from the CM. The velocity of P_i with respect to the translating reference frame is given by Eq. (2.12) as

$$\dot{\bar{r}}_i = (\dot{\bar{r}}_i)_{\text{body}} + \bar{\omega} \times \bar{r}_i$$

However, since the distance between P_i and CM is always constant, P_i has no velocity relative to the body axes, and the equation reduces to

$$\dot{\bar{r}} = \bar{\omega} \times \bar{r}_i$$

Hence, $\bar{H}$ (the angular momentum of the body about CM) is given by

$$\begin{aligned}\bar{H} &= \sum_i [\bar{r}_i \times m_i (\bar{\omega} \times \bar{r}_i)] \\ &= \sum_i m_i [r_i^2 \bar{\omega} - (\bar{r}_i \cdot \bar{\omega}) \bar{r}_i]\end{aligned}\quad (3.10)$$

Let us now define σ as the average density of the body. Then, the mass of the i^{th} particle can be written as

$$m_i = \sigma \Delta V_i \quad (3.11)$$

where ΔV_i is the volume element associated with the particle. Resolving $\bar{H}$ into components along the body axes, and, substituting Eq. (3.11) into Eq. (3.10), we have

$$\bar{H} = \begin{bmatrix} \left\{ \sum_i \left[r_i^2 p - (x_i p + y_i q + z_i r) x_i \right] \sigma \Delta V_i \right\} \\ \left\{ \sum_i \left[r_i^2 q - (x_i p + y_i q + z_i r) y_i \right] \sigma \Delta V_i \right\} \\ \left\{ \sum_i \left[r_i^2 r - (x_i p + y_i q + z_i r) z_i \right] \sigma \Delta V_i \right\} \end{bmatrix}$$

Grouping in terms of p , q , and r , and substituting $x_i^2 + y_i^2 + z_i^2$ for r_i^2 , we get

$$\bar{H} = \begin{bmatrix} \left[p \sum_i (y_i^2 + z_i^2) \sigma \Delta V_i & - q \sum_i x_i y_i \sigma \Delta V_i & - r \sum_i x_i z_i \sigma \Delta V_i \right] \\ \left[-p \sum_i y_i x_i \sigma \Delta V_i & + q \sum_i (x_i^2 + z_i^2) \sigma \Delta V_i & - r \sum_i y_i z_i \sigma \Delta V_i \right] \\ \left[-p \sum_i z_i x_i \sigma \Delta V_i & - q \sum_i z_i y_i \sigma \Delta V_i & + r \sum_i (x_i^2 + y_i^2) \sigma \Delta V_i \right] \end{bmatrix}$$

But the limit of

$$\sum_i (y_i^2 + z_i^2) \sigma \Delta V_i$$

as the maximum of the volume elements ΔV_i approaches zero is

$$\int_B (y^2 + z^2) \sigma dV$$

(this symbol represents the triple integral over the three-dimensional region defined by the rigid body). Since all the sums above can be replaced by integrals, we have

$$H = \begin{bmatrix} \left[p \int_B (y^2 + z^2) \sigma dV & - q \int_B x y \sigma dV & - r \int_B x z \sigma dV \right] \\ \left[-p \int_B y x \sigma dV & + q \int_B (x^2 + z^2) \sigma dV & - r \int_B y z \sigma dV \right] \\ \left[-p \int_B z x \sigma dV & - q \int_B z y \sigma dV & + r \int_B (x^2 + y^2) \sigma dV \right] \end{bmatrix}$$

Replacing the integral expression

$$\int_B (y^2 + z^2) \sigma \, dV$$

by the symbol I_{xx} , replacing the integral expression

$$\int_B x y \sigma \, dV$$

by I_{xy} , and replacing the remaining integrals in similar fashion, we get

$$\bar{H} = \begin{bmatrix} (p I_{xx} - q I_{xy} - r I_{xz}) \\ (-p I_{yx} + q I_{yy} - r I_{yz}) \\ (-p I_{zx} - q I_{zy} + r I_{zz}) \end{bmatrix} = H_x \bar{i} + H_y \bar{j} + H_z \bar{k} \quad (3.12)$$

Rewriting in matrix format, we have

$$\bar{H} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix} \quad (3.13)$$

I_{xx} , I_{yy} , and I_{zz} are called the moments of inertia of the body about the body axes X, Y, and Z, respectively. The other I terms are called the products of inertia.

The equations for the inertia terms are, of course, dependent on the geometric configuration of the body. The integral over the body with respect to the volume is replaced by the equivalent triple integrals written with respect to the body coordinate system. Quite frequently, spherical or cylindrical coordinates are used. Some inertia equations are obtained by semi-analytic, semi-empirical methods. In practice, the body coordinate axes are frequently selected so that they coincide with the principal directions at

the origin (i. e. , the three mutually perpendicular axes about which all products of inertia vanish). That such directions exist is given by a theorem in classical mechanics which will not be proved here. The moments of inertia taken about the principal axes are generally called the principal moments of inertia at the point selected as origin of the axes.

3.7 MOMENT EQUATIONS FOR BODY ORIGIN AT CENTER OF MASS.

3.7.1 So far, we have obtained detailed expressions for angular momentum and have shown that angular velocity is a component of angular momentum. The problem yet to be resolved is that of finding the relation between the torques of impressed forces and the resulting angular momentum.

Consider the equation of motion for each particle m_i of a rigid body:

$$m_i \ddot{\bar{r}}_i = \bar{F}_i \quad (3.14)$$

$\bar{F}_i$ is the sum of the forces acting on the i^{th} particle with position vector $\bar{r}_i$ from the origin O_0 of some inertial coordinate system. Taking the cross product of both sides of the equation with $\bar{r}_i$, we obtain

$$\bar{r}_i \times m_i \ddot{\bar{r}}_i = \bar{r}_i \times \bar{F}_i \quad (3.15)$$

But note that

$$\frac{d}{dt} (\bar{r}_i \times m_i \dot{\bar{r}}_i) = \dot{\bar{r}}_i \times m_i \ddot{\bar{r}}_i + \bar{r}_i \times m_i \dot{\dot{\bar{r}}_i} = \bar{r}_i \times m_i \ddot{\bar{r}}_i$$

Substituting in Eq. (3.15), there results the fundamental equation

$$\frac{d}{dt} (\bar{r}_i \times m_i \dot{\bar{r}}_i) = \bar{r}_i \times \bar{F}_i \quad (3.16)$$

Summing over all particles of the body,

$$\sum_i \frac{d}{dt} (\bar{\mathbf{r}}_i \times m_i \dot{\bar{\mathbf{r}}}_i) = \sum_i (\bar{\mathbf{r}}_i \times \bar{\mathbf{F}}_i)$$

thus

$$\sum_i \left(\frac{d\bar{\mathbf{H}}_i}{dt} \right)_{O_o} = \sum_i (\bar{\mathbf{L}}_i)_{O_o}$$

or

$$\frac{d\bar{\mathbf{H}}_{O_o}}{dt} = \bar{\mathbf{L}}_{O_o} \quad (3.17)$$

Equation (3.17) states that the time rate of change of the angular momentum of a rigid body about an inertial origin O_o is equal to the sum of all the torques impressed on the body.

The equations for angular momentum and torque are much more tractable if these quantities can be expressed about the center of mass. The question then arises: Does the rate of change of angular momentum about the mass center equal the impressed torque about the mass center? The answer: Yes; regardless of the fact that the center of mass is moving with respect to inertial axes (as we shall demonstrate).

Once again the axes X , Y , and Z (Fig. 35) form a Cartesian coordinate system attached to the rigid body B with origin taken as the center of mass CM . P_i is a particle of mass m_i at distance $\bar{\mathbf{R}}_i$ from the inertial origin O_o and at a distance $\bar{\mathbf{r}}_i$ from CM .

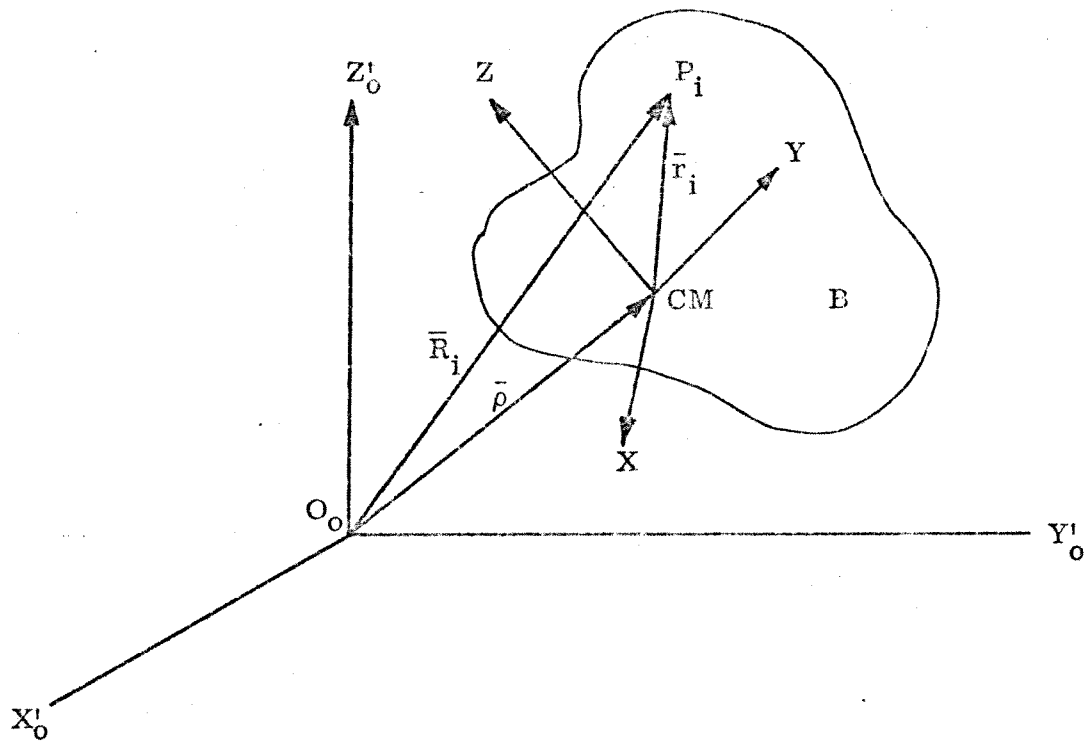


Fig. 35

Let $\bar{\rho}$ be the position vector of CM with respect to O_0 . Then, the angular momentum of B about O_0 [Eq. (3.9)] is

$$\begin{aligned}
 \bar{H}_{O_0} &= \sum_i (\bar{R}_i \times m_i \dot{\bar{R}}_i) \\
 &= \sum_i [(\bar{\rho} + \bar{r}_i) \times m_i (\dot{\bar{\rho}} + \dot{\bar{r}}_i)] \\
 &= \sum_i (\bar{\rho} \times m_i \dot{\bar{\rho}}) + \sum_i (\bar{\rho} \times m_i \dot{\bar{r}}_i) + \sum_i (\bar{r}_i \times m_i \dot{\bar{\rho}}) + \sum_i (\bar{r}_i \times m_i \dot{\bar{r}}_i) \\
 &= \bar{\rho} \times (\sum_i m_i) \dot{\bar{\rho}} + \bar{\rho} \times (\sum_i m_i \dot{\bar{r}}_i) + (\sum_i m_i \bar{r}_i) \times \dot{\bar{\rho}} + \sum_i (\bar{r}_i \times m_i \dot{\bar{r}}_i) \quad (3.18)
 \end{aligned}$$

But

$$\sum_i m_i \bar{\mathbf{r}}_i$$

gives the position of the center of mass with respect to the body origin. Since the body origin and the center of mass are coincident in this case,

$$\sum_i m_i \bar{\mathbf{r}}_i = 0$$

and hence,

$$\sum_i m_i \dot{\bar{\mathbf{r}}}_i = 0$$

Note, too, that

$$\sum_i (\bar{\mathbf{r}}_i \times m_i \dot{\bar{\mathbf{r}}}_i)$$

is by definition the angular momentum $\bar{\mathbf{H}}_{CM}$ of the body about its center of mass. Substituting these expressions into Eq. (3.18), we obtain

$$\bar{\mathbf{H}}_{O_o} = \bar{\boldsymbol{\rho}} \times m \dot{\bar{\boldsymbol{\rho}}} + \bar{\mathbf{H}}_{CM} \quad (3.19)$$

Differentiating Eq. (3.19),

$$\begin{aligned} \dot{\bar{\mathbf{H}}}_{O_o} &= \dot{\bar{\boldsymbol{\rho}}} \times m \dot{\bar{\boldsymbol{\rho}}} + \bar{\boldsymbol{\rho}} \times m \ddot{\bar{\boldsymbol{\rho}}} + \dot{\bar{\mathbf{H}}}_{CM} \\ \dot{\bar{\mathbf{H}}}_{O_o} &= \bar{\boldsymbol{\rho}} \times \sum_i \bar{\mathbf{F}}_i + \dot{\bar{\mathbf{H}}}_{CM} \end{aligned} \quad (3.20)$$

since

$$m\ddot{\rho} = \sum_i \bar{F}_i$$

by Newton's second law [Eq. (3.6)], where

$$\sum_i \bar{F}_i$$

is the sum of all the external forces acting on B. The torque $\bar{L}_{O_0}$ exerted by these forces about O_0 is

$$\begin{aligned}\bar{L}_{O_0} &= \sum_i (\bar{R}_i \times \bar{F}_i) \\ &= \sum_i [(\bar{\rho} + \bar{r}_i) \times \bar{F}_i] \\ &= \sum_i (\bar{\rho} \times \bar{F}_i) + \sum_i (\bar{r}_i \times \bar{F}_i)\end{aligned}$$

or

$$\bar{L}_{O_0} = \bar{\rho} \times \sum_i \bar{F}_i + \bar{L}_{CM} \quad (3.21)$$

where

$$\sum_i (\bar{r}_i \times \bar{F}_i)$$

is by definition the torque $\bar{L}_{CM}$ about the center of mass. But, since we showed with Eq. (3.17) that $\dot{\bar{H}}_{O_0} = \bar{L}_{O_0}$, we can substitute Eq. (3.20) into Eq. (3.21); from which, we verify that

$$\dot{\bar{H}}_{CM} = \bar{L}_{CM} \quad (3.22)$$

Thus, the rate of change of angular momentum of the body about the center of mass CM is equal to the sum of the torques about CM caused by the forces impressed on the body. For simplicity, we rewrite Eq. (3.22) as

$$\dot{\bar{H}} = \bar{L}$$

If $\bar{H}$ is measured with respect to the body axes as in the preceding section, we have

$$\dot{\bar{H}} = (\dot{\bar{H}})_{\text{body}} + \bar{\omega} \times \bar{H}$$

where $\bar{\omega}$ is the angular velocity of the body with respect to the translating reference frame. Expanding by components, we obtain

$$\begin{aligned} \dot{\bar{H}} &= \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ H_x & H_y & H_z \end{vmatrix} \\ &= (\dot{H}_x + H_z q - H_y r) \bar{i} + (\dot{H}_y + H_x r - H_z p) \bar{j} \\ &\quad + (\dot{H}_z + H_y p - H_x q) \bar{k} \end{aligned} \quad (3.23)$$

Let

$$\bar{L} = L_x \bar{i} + L_y \bar{j} + L_z \bar{k}$$

where L_x , L_y , and L_z are the sums of the torques about body axes X, Y, and Z, respectively.

since

$$m\ddot{\rho} = \sum_i \overline{F}_i$$

by Newton's second law [Eq. (3.6)] , where

$$\sum_i \overline{F}_i$$

is the sum of all the external forces acting on B . The torque $\overline{L}_{O_o}$ exerted by these forces about O_o is

$$\begin{aligned}\overline{L}_{O_o} &= \sum_i (\overline{R}_i \times \overline{F}_i) \\ &= \sum_i [(\overline{\rho} + \overline{r}_i) \times \overline{F}_i] \\ &= \sum_i (\overline{\rho} \times \overline{F}_i) + \sum_i (\overline{r}_i \times \overline{F}_i)\end{aligned}$$

or

$$\overline{L}_{O_o} = \overline{\rho} \times \sum_i \overline{F}_i + \overline{L}_{CM} \quad (3.21)$$

where

$$\sum_i (\overline{r}_i \times \overline{F}_i)$$

is by definition the torque $\overline{L}_{CM}$ about the center of mass. But, since we showed with Eq. (3.17) that $\dot{\overline{H}}_{O_o} = \overline{L}_{O_o}$, we can substitute Eq. (3.20) into Eq. (3.21); from which, we verify that

$$\dot{\overline{H}}_{CM} = \overline{L}_{CM} \quad (3.22)$$

Thus, the rate of change of angular momentum of the body about the center of mass CM is equal to the sum of the torques about CM caused by the forces impressed on the body. For simplicity, we rewrite Eq. (3.22) as

$$\dot{\bar{H}} = \bar{L}$$

If $\bar{H}$ is measured with respect to the body axes as in the preceding section, we have

$$\dot{\bar{H}} = (\dot{\bar{H}})_{\text{body}} + \bar{\omega} \times \bar{H}$$

where $\bar{\omega}$ is the angular velocity of the body with respect to the translating reference frame. Expanding by components, we obtain

$$\begin{aligned} \dot{\bar{H}} &= \dot{H}_x \bar{i} + \dot{H}_y \bar{j} + \dot{H}_z \bar{k} + \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ p & q & r \\ H_x & H_y & H_z \end{vmatrix} \\ &= (\dot{H}_x + H_z q - H_y r) \bar{i} + (\dot{H}_y + H_x r - H_z p) \bar{j} \\ &\quad + (\dot{H}_z + H_y p - H_x q) \bar{k} \end{aligned} \quad (3.23)$$

Let

$$\bar{L} = L_x \bar{i} + L_y \bar{j} + L_z \bar{k}$$

where L_x , L_y , and L_z are the sums of the torques about body axes X, Y, and Z, respectively.

By equating components $\bar{i}$, $\bar{j}$, and $\bar{k}$ of $\bar{L}$ and $\dot{\bar{H}}$ and substituting the expressions in Eq. (3.12) for H_x , H_y , H_z , ..., we obtain three equations:

$$L_x = \dot{p} I_{xx} - \dot{q} I_{xy} - \dot{r} I_{xz} + p \dot{i}_{xx} - q \dot{i}_{xy} - r \dot{i}_{xz} + q(-p I_{zx} - q I_{zy} + r I_{zz}) \\ - r(-p I_{yx} + q I_{yy} - r I_{yz})$$

$$L_y = -\dot{p} I_{yx} + \dot{q} I_{yy} - \dot{r} I_{yz} - p \dot{i}_{yz} + q \dot{i}_{yy} - r \dot{i}_{yz} + r(p I_{xx} - q I_{xy} - r I_{xz}) \\ - p(-p I_{zx} - q I_{zy} + r I_{zz})$$

$$L_z = -\dot{p} I_{zx} - \dot{q} I_{zy} + \dot{r} I_{zz} - p \dot{i}_{zx} - q \dot{i}_{zy} + r \dot{i}_{zz} + p(-p I_{yx} + q I_{yy} - r I_{yz}) \\ - q(p I_{xx} - q I_{xy} - r I_{xz})$$

With angular accelerations on the left, we rewrite the three equations (above) as

$$\dot{p} I_{xx} - \dot{q} I_{xy} - \dot{r} I_{xz} = L_x - (p \dot{i}_{xx} - q \dot{i}_{xy} - r \dot{i}_{xz}) - q(-p I_{zx} - q I_{zy} + r I_{zz}) \\ + r(-p I_{yx} + q I_{yy} - r I_{yz})$$

$$-\dot{p} I_{yx} + \dot{q} I_{yy} - \dot{r} I_{yz} = L_y - (p \dot{i}_{yz} + q \dot{i}_{yy} - r \dot{i}_{yz}) - r(p I_{xx} - q I_{xy} - r I_{xz}) \\ + p(-p I_{zx} - q I_{zy} + r I_{zz})$$

$$-\dot{p} I_{zx} - \dot{q} I_{zy} + \dot{r} I_{zz} = L_z - (-p \dot{i}_{zx} - q \dot{i}_{zy} + r \dot{i}_{zz}) - p(-p I_{yx} + q I_{yy} - r I_{yz}) \\ + q(p I_{xx} - q I_{xy} - r I_{xz})$$

(3.25)

These three equations are known as the moment equations of the rigid body for the case in which the center of mass is the body-axes origin. These equations can be solved simultaneously to obtain algebraic expressions for $\dot{p}$, $\dot{q}$, and $\dot{r}$, which can then be

integrated to obtain the angular velocities p , q , and r . The derivatives of the direction cosines relating the body coordinate system to the translating reference frame can then be computed as shown in paragraph 3.8.1.

If the body possesses certain geometric properties, the moment equations become simplified. For example, if the body is symmetric with respect to all its axes, the product of inertia terms and their derivatives vanish. An interesting analog to Newton's second law of motion results for a symmetric body of constant mass rotating with constant velocity about only one of its axes. Let us say that this body is rotating with constant angular velocity p about its X axis; q and r are zero. Because the mass is constant, $\dot{I}_{xx}$ is also zero. The moment equations reduce to one equation:

$$I_{xx}\dot{p} = L_x$$

Note the similarity, in this case, to motion of the center of mass in the X direction:

$$ma_x = F_x$$

3.8 CHANGE OF BODY ORIENTATION.

3.8.1 So far, we have obtained this relationship between the impressed forces and the angular motion of a body. We shall now show how the body orientation varies with the angular velocity. As stated in paragraph 3.2.1, the instantaneous orientation of the body with respect to the translating reference frame (X_o , Y_o , Z_o) may be given by the orientation of a Cartesian coordinate system (X , Y , Z) to which the body is rigidly attached. In this case, the origin of the body axes X , Y , and Z is taken at the body center of mass.

As developed in paragraph 1.4.1, the orientation is specified by an orthogonal transformation between the two coordinate systems. The transformation is composed of nine direction cosines which can be computed from three independent angles of rotation known

as the Euler angles. The orientation problem, then, is to determine how the transformation varies with time as a result of the angular motion of the body. This can be determined by integrating the time rates of change of either the Euler angles or of the direction cosines; for reasons stated at the end of this section (3.8), it is the direction cosine approach which is developed.

Before proceeding further, however, we need to expand upon the relation between the time derivative of a vector $\bar{A}$ as measured in one coordinate system (which will be called the reference system) and the derivative of the same vector as measured in a coordinate system moving relative to the reference frame. This relationship is developed in paragraph 2.11.1 and stated in Eq. (2.14), which we shall write in the more general form

$$\left(\frac{d\bar{A}}{dt}\right)_{\text{reference}} = \left(\frac{d\bar{A}}{dt}\right)_{\text{relative}} + \dot{\bar{\rho}} + \bar{\omega} \times \bar{A} \quad (3.26)$$

where

$$\begin{aligned} \left(\frac{d\bar{A}}{dt}\right)_{\text{reference}} &= \text{the derivative of } \bar{A} \text{ as seen by an observer in the reference coordinate frame} \\ \left(\frac{d\bar{A}}{dt}\right)_{\text{relative}} &= \text{the derivative of } \bar{A} \text{ as seen by an observer in the frame moving relative to the reference frame} \\ \dot{\bar{\rho}} &= \text{the velocity of the origin of the moving (or "relative") frame} \\ \bar{\omega} &= \text{the angular velocity of the moving frame relative to the reference system.} \end{aligned}$$

Note that, if the relative frame is only rotating and not translating, the velocity of its origin will be zero, and Eq. (3.26) becomes

$$\left(\frac{d\bar{A}}{dt}\right)_{\text{reference}} = \left(\frac{d\bar{A}}{dt}\right)_{\text{relative}} + \bar{\omega} \times \bar{A} \quad (3.27)$$

The reference system in Eqs. (3.26) and (3.27) is considered fixed in space only as seen by an observer in the relative system. In point of fact, the reference system may be moving with respect to a third coordinate system, but Eq. (3.27) is still a valid expression which relates one coordinate system that is rotating with respect to another. Suppose that the reference system is only translating with respect to an inertial system and that its axes remain parallel to the corresponding inertial axes: Then, the orientation of the relative system (X, Y, Z) with respect to the inertial system (X₀, Y₀, Z₀) is identical to the orientation of (X, Y, Z) with respect to the reference system (X₀, Y₀, Z₀). In our previous discussion, we called frame (X₀, Y₀, Z₀) the translating reference frame and actually stipulated that its origin coincide with the origin of the body axes X, Y, and Z. Under these conditions, Eq. (3.27) gives the derivative of $\bar{A}$ with respect to the translating reference system; adding the velocity of the common origin, we obtain the total derivative of $\bar{A}$ with respect to the inertial frame [Eq. (3.26)].

We return now to the problem of finding the instantaneous orientation of the body axes with respect to the translating reference frame. Let $\bar{i}_0, \bar{j}_0, \bar{k}_0$ be the orthonormal vectors of the translating reference system, and let $\bar{i}, \bar{j}, \bar{k}$ be the orthonormal vectors of the body-fixed coordinate system. As shown in paragraph 1.4.1, these two sets of vectors are related by a matrix of direction cosines, which transforms coordinates in body axes to coordinates in reference axes:

$$\begin{bmatrix} \bar{i}_0 \\ \bar{j}_0 \\ \bar{k}_0 \end{bmatrix} = \begin{bmatrix} C_{xx} & C_{yx} & C_{zx} \\ C_{xy} & C_{yy} & C_{zy} \\ C_{xz} & C_{yz} & C_{zz} \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \quad (3.28)$$

Suppose that the body or "relative" coordinate system is rotating with angular velocity $\bar{\omega} = p \bar{i} + q \bar{j} + r \bar{k}$ with respect to the translating reference system (X_o, Y_o, Z_o) . Then Eq. (3.27) can be used to obtain the derivatives of $\bar{i}_o$, $\bar{j}_o$, and $\bar{k}_o$. Thus,

$$\left(\frac{d\bar{i}_o}{dt}\right)_{\text{reference}} = \left(\frac{d\bar{i}_o}{dt}\right)_{\text{body}} + \bar{\omega} \times \bar{i}_o$$

$$= \left(\frac{d\bar{i}_o}{dt}\right)_{\text{body}} - \bar{i}_o \times \bar{\omega}$$

$$= 0 \quad \text{since } \bar{i}_o \text{ is fixed in magnitude and direction in the reference system.}$$

Therefore,

$$\left(\frac{d\bar{i}_o}{dt}\right)_{\text{body}} = \bar{i}_o \times \bar{\omega} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ C_{xx} & C_{yx} & C_{zx} \\ p & q & r \end{vmatrix}$$

or

$$\left(\frac{d\bar{i}_o}{dt}\right)_{\text{body}} = (rC_{yx} - qC_{zx})\bar{i} + (pC_{zx} - rC_{xx})\bar{j} + (qC_{xx} - pC_{yx})\bar{k} \quad (3.29)$$

Similarly,

$$\left(\frac{d\bar{j}_o}{dt}\right)_{\text{body}} = (rC_{zy} - qC_{xy})\bar{i} + (pC_{xy} - rC_{yy})\bar{j} + (qC_{yy} - pC_{zy})\bar{k} \quad (3.30)$$

and

$$\left(\frac{d\bar{k}_o}{dt}\right)_{\text{body}} = (rC_{yz} - qC_{zz})\bar{i} + (pC_{zz} - rC_{xz})\bar{j} + (qC_{xz} - pC_{yz})\bar{k} \quad (3.31)$$

Arranging Eqs. (3.29), (3.30), and (3.31) in matrix format, we have

$$\begin{bmatrix} \frac{d\bar{i}_o}{dt} \\ \frac{d\bar{j}_o}{dt} \\ \frac{d\bar{k}_o}{dt} \end{bmatrix}_{\text{body}} = \begin{bmatrix} [r C_{yx} - q C_{zx}] & [p C_{zx} - r C_{xx}] & [q C_{xx} - p C_{yx}] \\ [r C_{yy} - q C_{zy}] & [p C_{zy} - r C_{xy}] & [q C_{xy} - p C_{yy}] \\ [r C_{yz} - q C_{zz}] & [p C_{zz} - r C_{xz}] & [q C_{xz} - p C_{yz}] \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \quad (3.32)$$

But by straightforward differentiation of Eq. (3.28), we obtain

$$\begin{bmatrix} \frac{d\bar{i}_o}{dt} \\ \frac{d\bar{j}_o}{dt} \\ \frac{d\bar{k}_o}{dt} \end{bmatrix}_{\text{body}} = \begin{bmatrix} \dot{C}_{xx} & \dot{C}_{yx} & \dot{C}_{zx} \\ \dot{C}_{xy} & \dot{C}_{yy} & \dot{C}_{zy} \\ \dot{C}_{xz} & \dot{C}_{yz} & \dot{C}_{zz} \end{bmatrix} \begin{bmatrix} \bar{i} \\ \bar{j} \\ \bar{k} \end{bmatrix} \quad (3.33)$$

Note that in differentiating Eq. (3.28), the terms involving

$$\left(\frac{d\bar{i}}{dt}\right)_{\text{body}}, \quad \left(\frac{d\bar{j}}{dt}\right)_{\text{body}}, \quad \left(\frac{d\bar{k}}{dt}\right)_{\text{body}}$$

vanish since these derivatives are zero when $\bar{i}$, $\bar{j}$, $\bar{k}$ are fixed with respect to the body axes.

Thus, by equating the matrices of Eqs. (3.32) and (3.33), we obtain a set of nine differential equations, which, when integrated, will furnish the direction cosines that give the orientation of the body coordinate system with respect to the reference frame. That is, for $\dot{C}_{xx}$ and $\dot{C}_{zy}$ (as an example), we obtain

$$\dot{C}_{xx} = \frac{dC_{xx}}{dt} = r C_{yx} - q C_{xx}$$

$$\dot{C}_{zy} = \frac{dC_{zy}}{dt} = q C_{xy} - p C_{yy}$$

The direction cosine differential equations are thus functions of the direction cosines and the body angular velocity. They are continuous functions, whereas the Euler angle differential equations (not derived here) are discontinuous and less tractable. It also has been found a superior method (from a numerical computation viewpoint) to obtain the transformation by integrating the direction cosine derivatives rather than by integrating the Euler angle derivatives and then computing the transformation by the methods shown in Section 1.4. Errors obtained in the numerical integration of the Euler angle rates may be compounded by roundoff errors and slight inaccuracies in the trigonometric calculations that are required to obtain the transformation from the Euler angles.

The direction cosine derivatives are smoother time functions than the Euler angle derivatives and, hence, can be numerically integrated with greater accuracy. The properties of the direction cosine matrix [the dot product of one row (or column) with a different row (or column) being zero, and the dot product of one row (or column) with itself being equal to one, as discussed in Section 1.4.1] can be used to check the accuracy of the integration or to reduce the number of equations being integrated.

3.9 MOMENT EQUATIONS FOR ARBITRARY BODY ORIGIN.

3.9.1 It is important to remember that the moment equations derived in Section 3.7 are valid only when torques are taken about the center of mass. The following discussion develops the moment equations for the case in which torques are taken about a body origin which is a point of the body not at the center of mass. Once again, we employ the translating reference frame (X_o , Y_o , Z_o) whose origin O coincides with that of the body axes X , Y , and Z . Let the position vector of the center of mass with respect to O be $\bar{r}_{CM}$. P_i is an arbitrary point of the body at distance $\bar{R}_i$ from O and distance $\bar{r}_i$ from CM .

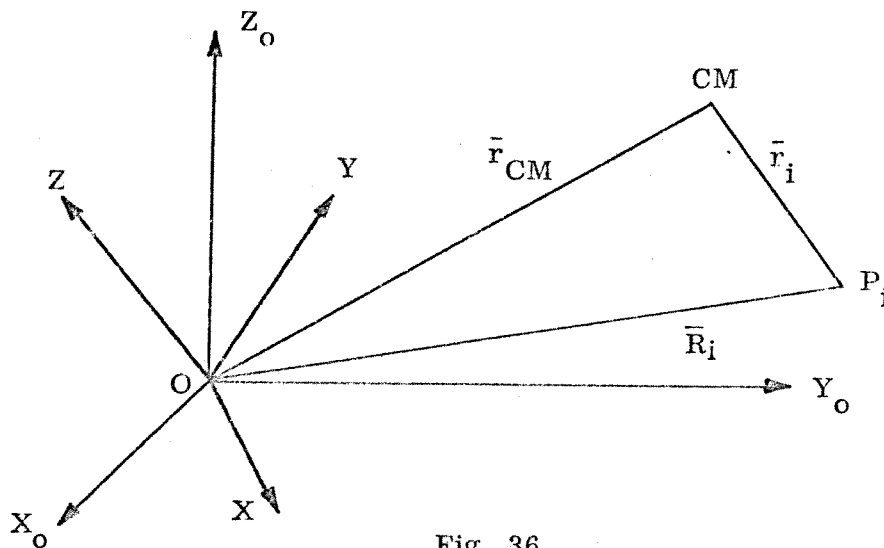


Fig. 36

The angular momentum of the body about O is

$$\begin{aligned}
 \bar{H}_O &= \sum_i (\bar{R}_i \times m_i \dot{\bar{R}}_i) \\
 &= \sum_i [(\bar{r}_{CM} + \bar{r}_i) \times m_i (\dot{\bar{r}}_{CM} + \dot{\bar{r}}_i)] \\
 &= \bar{r}_{CM} \times m \dot{\bar{r}}_{CM} + \bar{r}_{CM} \times \left(\sum_i m_i \dot{\bar{r}}_i \right) + \left(\sum_i m_i \bar{r}_i \right) \times \dot{\bar{r}}_{CM} + \sum_i (\bar{r}_i \times m_i \dot{\bar{r}}_i)
 \end{aligned}
 \tag{3.34}$$

But,

$$\sum_i m_i \bar{\mathbf{r}}_i = 0 \quad \text{and} \quad \sum_i m_i \dot{\bar{\mathbf{r}}} = 0$$

by definition of the center of mass [Eq. (3.5)], and

$$\sum_i (\bar{\mathbf{r}}_i \times m_i \dot{\bar{\mathbf{r}}}_i)$$

is the angular momentum $\bar{\mathbf{H}}_{\text{CM}}$ of the body about the center of mass. Substituting these relations into Eq. (3.34) and solving for $\bar{\mathbf{H}}_{\text{CM}}$, we obtain

$$\bar{\mathbf{H}}_{\text{CM}} = \bar{\mathbf{H}}_O - \bar{\mathbf{r}}_{\text{CM}} \times m \dot{\bar{\mathbf{r}}}_{\text{CM}}$$

Therefore,

$$\dot{\bar{\mathbf{H}}}_{\text{CM}} = \dot{\bar{\mathbf{H}}}_O - \bar{\mathbf{r}}_{\text{CM}} \times m \ddot{\bar{\mathbf{r}}}_{\text{CM}} \quad (3.35)$$

The torque of the body about O is

$$\begin{aligned} \bar{\mathbf{L}}_O &= \sum_i [(\bar{\mathbf{r}}_{\text{CM}} + \bar{\mathbf{r}}_i) \times \bar{\mathbf{F}}_i] \\ &= \bar{\mathbf{r}}_{\text{CM}} \times \sum_i \bar{\mathbf{F}}_i + \sum_i (\bar{\mathbf{r}}_i \times \bar{\mathbf{F}}_i) \end{aligned}$$

or

$$\bar{\mathbf{L}}_O = \bar{\mathbf{r}}_{\text{CM}} \times \bar{\mathbf{F}} + \bar{\mathbf{L}}_{\text{CM}}$$

where $\bar{\mathbf{F}}$ is the sum of all forces.

Therefore,

$$\bar{L}_{CM} = \bar{L}_O - \bar{r}_{CM} \times \bar{F} \quad (3.36)$$

Since $\dot{\bar{H}}_{CM} = \dot{\bar{L}}_{CM}$, the right-hand sides of Eqs. (3.35) and (3.36) are equal, giving

$$\dot{\bar{H}}_O - \bar{r}_{CM} \times \ddot{\bar{r}}_{CM} = \bar{L}_O - \bar{r}_{CM} \times \bar{F} \quad (3.37)$$

This expression [Eq. (3.37)] is the moment equation for the case in which forces and torques are measured about a body origin not at the center of mass; it is valid, of course, even when the center of mass is moving with respect to the body coordinate system.

In order to write the detailed component moment equations along the body axes, we need to find an expression for $\ddot{\bar{r}}_{CM}$ (the acceleration of the mass center with respect to the translating reference frame centered at O). For this purpose, we use Eq. (2.15), which becomes

$$\ddot{\bar{r}}_{CM} = (\ddot{\bar{r}}_{CM})_{body} + \dot{\bar{\omega}} \times \bar{r}_{CM} + 2\bar{\omega} \times (\dot{\bar{r}}_{CM})_{body} + \bar{\omega} \times (\bar{\omega} \times \bar{r}_{CM}) \quad (3.38)$$

Since the origins of the body axes and the reference frame coincide, the acceleration $\ddot{\bar{p}}$ [Eq. (2.15)] of the body origin with respect to the translating reference frame is zero. The angular velocity of the body with respect to the reference frame is

$$\bar{\omega} = p \bar{i} + q \bar{j} + r \bar{k}$$

where $\bar{i}$, $\bar{j}$, and $\bar{k}$ are the orthonormal vectors of the body axes. The instantaneous distance of the center of mass from O is

$$\bar{r}_{CM} = x_c \bar{i} + y_c \bar{j} + z_c \bar{k}$$

We now express each of the terms in Eq. (3.38) by components along the body axes:

$$(\ddot{\bar{r}}_{CM})_{body} = \ddot{x}_c \bar{i} + \ddot{y}_c \bar{j} + \ddot{z}_c \bar{k} \quad (3.39)$$

$$\dot{\bar{\omega}} \times \bar{r}_{CM} = (\dot{q}z_c - \dot{r}y_c)\bar{i} + (\dot{r}x_c - \dot{p}z_c)\bar{j} + (\dot{p}y_c - \dot{q}x_c)\bar{k} \quad (3.40)$$

$$2\bar{\omega} \times (\dot{\bar{r}}_{CM})_{body} = 2(q\dot{z}_c - r\dot{y}_c)\bar{i} + 2(r\dot{x}_c - p\dot{z}_c)\bar{j} + 2(p\dot{y}_c - q\dot{x}_c)\bar{k} \quad (3.41)$$

$$\bar{\omega} \times (\bar{\omega} \times \bar{r}_{CM}) = \begin{bmatrix} (px_c + qy_c + rz_c)p - \omega^2 x_c \\ (px_c + qy_c + rz_c)q - \omega^2 y_c \\ (px_c + qy_c + rz_c)r - \omega^2 z_c \end{bmatrix} \quad (3.42)$$

Substituting these expressions into Eq. (3.38), we obtain

$$\ddot{\bar{r}}_{CM} = \begin{bmatrix} \ddot{x}_c + (\dot{q}z_c - \dot{r}y_c) + 2(q\dot{z}_c - r\dot{y}_c) + (px_c + qy_c + rz_c)p - \omega^2 x_c \\ \ddot{y}_c + (\dot{r}x_c - \dot{p}z_c) + 2(r\dot{x}_c - p\dot{z}_c) + (px_c + qy_c + rz_c)q - \omega^2 y_c \\ \ddot{z}_c + (\dot{p}y_c - \dot{q}x_c) + 2(p\dot{y}_c - q\dot{x}_c) + (px_c + qy_c + rz_c)r - \omega^2 z_c \end{bmatrix} \quad (3.43)$$

The second term on the right-hand side of Eq. (3.37) becomes

$$\bar{r}_{CM} \times \bar{F} = (y_c F_z - z_c F_y)\bar{i} + (z_c F_x - x_c F_z)\bar{j} + (x_c F_y - y_c F_x)\bar{k} \quad (3.44)$$

The three expressions in Eq. (3.25) represent the moment equations when $\dot{\bar{H}}_O = \bar{L}_O$. Thus, we need to add the appropriate components of $\bar{r}_{CM} \times m\ddot{\bar{r}}_{CM} - \bar{r}_{CM} \times \bar{F}$ to

the right-hand sides of Eq. (3.25) in order to obtain Eq. (3.37). This we will do to the first expression in Eq. (3.25), which represents the $\bar{i}$ component of Eq. (3.37):

$$\begin{aligned} \dot{p}I_{xx} - \dot{q}I_{xy} - \dot{r}I_{xz} = & L_x - (p\dot{I}_{xx} - q\dot{I}_{xy} - r\dot{I}_{xz}) - q(-pI_{zx} - qI_{zy} + rI_{zz}) \\ & + r(-pI_{yx} + qI_{yy} - rI_{yz}) + y_c[m\ddot{z}_c + (\dot{p}y_c - \dot{q}x_c) \\ & + 2(\dot{p}y_c - \dot{q}x_c) + (px_c + qy_c + rz_c)r - \omega^2 z_c] \\ & - z_c[m\ddot{y}_c + (\dot{r}x_c - \dot{p}z_c) + 2(r\dot{x}_c - p\dot{z}_c) + (px_c + qy_c \\ & + rz_c)q - \omega^2 y_c] - (y_c F_z - z_c F_y) \end{aligned}$$

Collecting all $\dot{p}$, $\dot{q}$, and $\dot{r}$ terms and rearranging certain others, we obtain

$$\begin{aligned} & \dot{p}[I_{xx} - m(y_c^2 + z_c^2)] - \dot{q}(I_{xy} - mx_c y_c) - \dot{r}(I_{xz} - mx_c z_c) \quad (3.45) \\ = & L_x - (p\dot{I}_{xx} - q\dot{I}_{xy} - r\dot{I}_{xz}) - q(-pI_{zx} - qI_{zy} + rI_{zz}) + r(-pI_{yx} + qI_{yy} - rI_{yz}) \\ & + y_c m[\ddot{z}_c + 2(\dot{p}y_c - \dot{q}x_c) + r(px_c + qy_c + rz_c)] - z_c m[\ddot{y}_c + 2(r\dot{x}_c - p\dot{z}_c) \\ & + q(px_c + qy_c + rz_c)] - (y_c F_z - z_c F_y) \end{aligned}$$

Performing the similar operations on the 2nd and 3rd expressions of Eq. (3.25) we obtain the $\bar{j}$ and $\bar{k}$ components of Eq. (3.37):

$$\begin{aligned} & -\dot{p}(I_{xy} - mx_c y_c) + \dot{q}[I_{yy} - m(x_c^2 + z_c^2)] - \dot{r}(I_{yz} - my_c z_c) \quad (3.46) \\ = & L_y - (-p\dot{I}_{yz} + q\dot{I}_{yy} - r\dot{I}_{yz}) - r(pI_{xx} - qI_{xy} - rI_{xz}) + p(-pI_{zx} - qI_{zy} + rI_{zz}) \\ & + z_c m[\ddot{x}_c + 2(q\dot{z}_c - r\dot{y}_c) + p(px_c + qy_c + rz_c)] - x_c m[\ddot{z}_c + 2(\dot{p}y_c - \dot{q}x_c) \\ & + r(px_c + qy_c + rz_c)] - (z_c F_x - x_c F_z) \end{aligned}$$

$$\begin{aligned}
& -\dot{p}(I_{xz} - m x_c z_c) - \dot{q}(I_{yz} - m y_c z_c) + \dot{r}[I_{zz} - m(x_c^2 + y_c^2)] \quad (3.47) \\
& = L_z - (-p\dot{I}_{zx} - q\dot{I}_{zy} + r\dot{I}_{zz}) - p(-pI_{yx} + qI_{yy} - rI_{yz}) + q(pI_{xx} - qI_{xy} - rI_{xz}) \\
& \quad + x_c m [\ddot{y}_c + 2(r\dot{x}_c - p\dot{z}_c) + q(px_c + qy_c + rz_c)] - y_c m [\ddot{x}_c + 2(q\dot{z}_c - r\dot{y}_c) \\
& \quad + p(px_c + qy_c + rz_c)] - (x_c F_y - y_c F_x)
\end{aligned}$$

Equations (3.45), (3.46), and (3.47) are the complete moment equations for the case in which forces and torques are measured about a body-axes origin not at the center of mass. It is apparent that the moment and product-of-inertia terms on these equations must be computed with respect to the body origin rather than the center of mass. No general expressions can be written for the position, velocity, and acceleration of the mass center relative to the body origin (that is, for: x_c, y_c, z_c ; $\dot{x}_c, \dot{y}_c, \dot{z}_c$; and $\ddot{x}_c, \ddot{y}_c, \ddot{z}_c$) as these quantities are functions of the specific body configuration and motion to be studied.

3.10 ACCELERATION OF A POINT FIXED IN THE BODY.

3.10.1 In applications, it is frequently necessary to compute the acceleration of a body point with respect to the center of mass, where the distance $\bar{r}$ (between point P and the mass center CM) remains constant. Equation (2.15), once again, gives the correct expression except that, in this case, the velocity $\bar{v}$ and acceleration $\bar{a}$ of the point with respect to the center of mass are zero, since the distance between P and CM is constant. Since the origins of the body axes and the reference axes coincide, the acceleration $\ddot{\rho}$ of the body origin with respect to the translating reference frame is zero. So, Eq. (2.15) becomes

$$\ddot{\bar{r}} = \dot{\bar{\omega}} \times \bar{r} + \bar{\omega} \times (\bar{\omega} \times \bar{r})$$

Letting

$$\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k} \quad \text{and} \quad \bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$$

then

$$\ddot{\bar{r}} = \begin{bmatrix} \dot{q}z - \dot{r}y + (px + qy + rz)p - (p^2 + q^2 + r^2)x \\ \dot{r}x - \dot{p}z + (px + qy + rz)q - (p^2 + q^2 + r^2)y \\ \dot{p}y - \dot{q}x + (px + qy + rz)r - (p^2 + q^2 + r^2)z \end{bmatrix}$$

or

$$\ddot{\bar{r}} = \begin{bmatrix} (-q^2 - r^2)x + (-\dot{r} + pq)y + (\dot{q} + pr)z \\ (r + pq)x + (-r^2 - p^2)y + (-\dot{p} + qr)z \\ (-q + pr)x + (\dot{p} + qr)y + (-p^2 - q^2)z \end{bmatrix} \quad (3.48)$$

To obtain the total acceleration of P , one must add to this the acceleration of the center of mass with respect to the inertial coordinate system being used.

3.11 EULER ANGLE DERIVATIVES.

3.11.1 As discussed in Section 3.8, the time variant transformation matrix of rotation between two coordinate systems can be obtained by numerically integrating the time rates of change of the direction cosines. It is also possible to integrate the time derivatives of the Euler angles and then obtain the transformation matrix directly from the angles, as shown in Section 1.4.2. This procedure is generally not followed, as explained in Section 3.8. For the sake of completeness, however, the time rates of change of Euler angles are derived below.

Let (X_0, Y_0, Z_0) be a reference frame and let (X, Y, Z) be another reference frame obtained from (X_0, Y_0, Z_0) by three successive rotations. For this example, let us take the displacements to be a roll ϕ , followed by a pitch θ , and, finally, by yaw ψ . Associated with each angular displacement is an angular velocity $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$, respectively. If a rigid body is affixed to the coordinate system (X, Y, Z) , the body will have an angular velocity $\bar{\omega}$ equal to the vectorial sum of the three simultaneous velocities $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$; i.e.,

$$\bar{\omega} = \dot{\phi} + \dot{\theta} + \dot{\psi}$$

roll pitch yaw

However, the body angular velocity is normally obtained by integrating the moment equations to obtain p , q , and r (the components of $\bar{\omega}$ along each of the X, Y, Z axes). Thus, we have the relation

$$\bar{\omega} = p\bar{i} + q\bar{j} + r\bar{k} = \dot{\phi} + \dot{\theta} + \dot{\psi}$$

By taking the components of $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ along the body axes, we have expressions for obtaining $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$ in terms of p, q, r, ϕ, θ , and ψ . Let us now examine the three successive rotations as illustrated in Fig. 37.

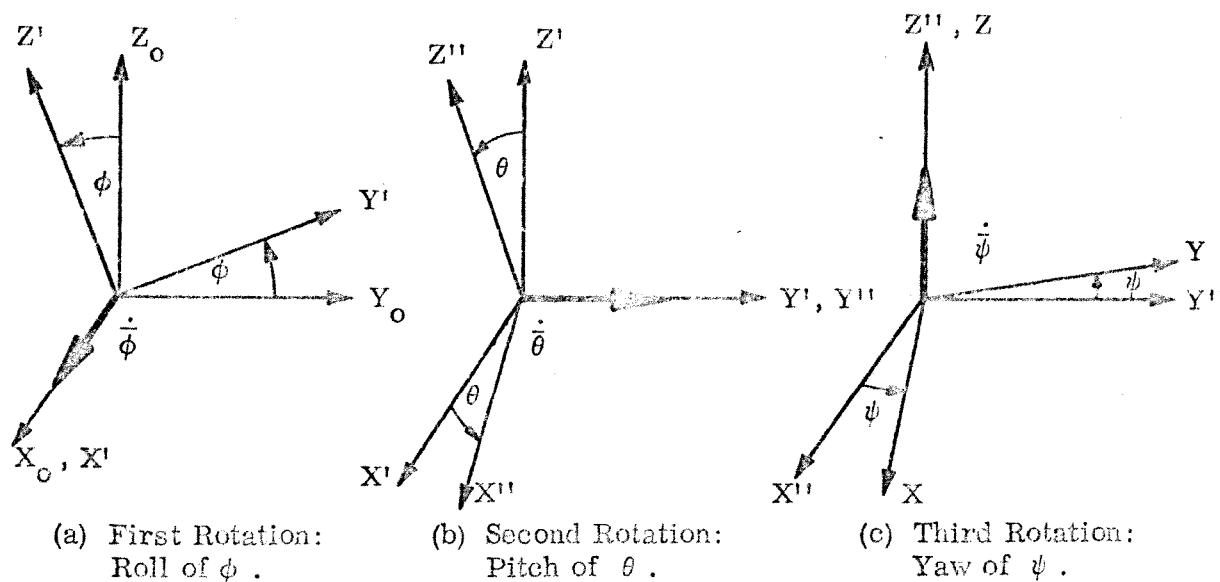


Fig. 37

The angular velocity $\dot{\phi}$ (due to rotation ϕ) is normal to the Y_0-Z_0 plane and falls along X_0 . The components of $\dot{\phi}$ in coordinate system (X, Y, Z) are obtained by applying the transformation from frame (X_0, Y_0, Z_0) to frame (X, Y, Z) . Thus,

$$\dot{\phi}_{X, Y, Z \text{ coords.}} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} \dot{\phi} \\ 0 \\ 0 \end{bmatrix} \quad (3.49)$$

$$= \begin{bmatrix} \dot{\phi} \cos \psi \cos \theta \\ -\dot{\phi} \sin \psi \cos \theta \\ \dot{\phi} \sin \theta \end{bmatrix}$$

The angular velocity $\dot{\theta}$ (due to rotation θ) is normal to the $X'-Z'$ plane and falls along Y' . The components of $\dot{\theta}$ in system (X, Y, Z) are obtained by transformation from coordinates $X', Y',$ and Z' :

$$\dot{\theta}_{X, Y, Z \text{ coords.}} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} 0 \\ \dot{\theta} \\ 0 \end{bmatrix} \quad (3.50)$$

$$= \begin{bmatrix} \dot{\theta} \sin \psi \\ \dot{\theta} \cos \psi \\ 0 \end{bmatrix}$$

The angular velocity $\dot{\psi}$ (due to rotation ψ) is normal to the $X''-Y''$ plane and falls along Z'' . The components of $\dot{\psi}$ in system (X, Y, Z) are obtained from coordinates $X'', Y'',$ and Z'' .

$$\dot{\psi}_{X, Y, Z \text{ coords.}} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \dot{\psi} \end{bmatrix} \quad (3.51)$$

Finally, the total angular velocity $\bar{\omega}$ is obtained as the sum of Eqs. (3.49), (3.50), and (3.51):

$$\bar{\omega} \equiv \begin{bmatrix} p \\ q \\ r \end{bmatrix} = \begin{bmatrix} \dot{\phi} \cos \psi \cos \theta + \dot{\theta} \sin \psi \\ -\dot{\phi} \sin \psi \cos \theta + \dot{\theta} \cos \psi \\ \dot{\phi} \sin \theta + \dot{\psi} \end{bmatrix} \quad (3.52)$$

This system of equations can be solved simultaneously for $\dot{\phi}$, $\dot{\theta}$, and $\dot{\psi}$, yielding

$$\begin{aligned} \dot{\phi} &= \frac{p \cos \psi - q \sin \psi}{\cos \theta} \\ \dot{\theta} &= p \sin \psi + q \cos \psi \\ \dot{\psi} &= r - \dot{\phi} \sin \theta \end{aligned} \quad (3.53)$$

Numerical integration of these time rates of change will give the Euler angles for rotations in the order roll, pitch, and yaw. For different orders of rotation, different derivative equations from those in Eq. (3.53) will be obtained.